

# More Properties of the Ramanujan Sequence

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## ABSTRACT

The Ramanujan sequence  $\{\theta_n\}_{n \geq 0}$ , defined as

$$\theta_0 = \frac{1}{2}, \quad \theta_n = \left( \frac{e^n}{2} - \sum_{k=0}^{n-1} \frac{n^k}{k!} \right) \cdot \frac{n!}{n^n}, \quad n \geq 1,$$

has been studied on many occasions and in many different contexts. J. Adell and P. Jodra [2](2008) and S. Koumandos [6](2013) showed, respectively, that the sequences  $\{\theta_n\}_{n \geq 0}$  and  $\{4/135 - n \cdot (\theta_n - 1/3)\}_{n \geq 0}$  are completely monotone. In the present paper we establish that the sequence  $\{(n+1) \cdot (\theta_n - 1/3)\}_{n \geq 0}$  is also completely monotone. Furthermore, we prove that the analytic function  $(\theta_1 - 1/3)^{-1} \sum_{n=1}^{\infty} (\theta_n - 1/3) \cdot z^n / n^\alpha$  is universally starlike for every  $\alpha \geq 1$  in the slit domain  $\mathbb{C} \setminus [1, \infty)$ . This seems to be the first result putting the Ramanujan sequence into the context of analytic univalent functions and is a step towards a previous stronger conjecture, proposed by S. Ruscheweyh, L. Salinas and T. Sugawa in [10](2009), namely that the function  $(\theta_1 - 1/3)^{-1} \sum_{n=1}^{\infty} (\theta_n - 1/3) \cdot z^n$  is universally convex.

## 1. Introduction

A famous problem raised by Ramanujan in [8](1911) states that the so-called Ramanujan numbers  $\theta_n$ ,  $n \geq 0$ , defined as

$$\theta_0 = \frac{1}{2}, \quad \theta_n = \left( \frac{e^n}{2} - \sum_{k=0}^{n-1} \frac{n^k}{k!} \right) \cdot \frac{n!}{n^n}, \quad n \geq 1, \quad (1.1)$$

satisfy

$$\theta_n \in \left[ \frac{1}{3}, \frac{1}{2} \right]. \quad (1.2)$$

In his first letter to Hardy [8](1913) Ramanujan refined his conjecture (1.2) as follows

$$\theta_n = \frac{1}{3} + \frac{4}{135} \cdot \frac{1}{n + k_n}, \quad k_n \in \left[ \frac{2}{21}, \frac{8}{45} \right], \quad n \geq 0. \quad (1.3)$$

The first proofs of (1.2) were published by G.Szegő [11] (1928) and G.N.Watson [12](1929). A proof of (1.3) was obtained in 1995 by Flajolet et al. [5].

In 2003 S.E.Alm [1] showed that the sequence  $\{k_n\}_{n \geq 0}$  appearing in (1.3) is decreasing. In 2008 J. Adell and P. Jodra [2] proved that there is a probability distribution function  $F$  on  $[0, 1]$  such that

$$\theta_n - 1/3 = \frac{1}{6} \int_0^1 x^n dF(x), \quad n \geq 0, \quad (1.4)$$

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which implies that the sequence  $\{\theta_n\}_{n \geq 0}$  is completely monotone, i.e.

$$\sum_{m=0}^n \binom{n}{m} (-1)^m \theta_{k+m} \geq 0, \quad k \geq 0, \quad n \geq 0. \quad (1.5)$$

In 2013 S. Koumandos [6] proved the existence of a strictly positive function  $k$  on  $[0, +\infty)$  such that

$$\frac{4}{135} - n \cdot \left( \theta_n - \frac{1}{3} \right) = \frac{1}{2} \int_0^\infty e^{-nx} k(x) dx, \quad n \geq 0, \quad (1.6)$$

and noted that the complete monotony of the sequence  $\{4/135 - n \cdot (\theta_n - 1/3)\}_{n \geq 0}$  follows from (1.6).

We refer the reader to Alzer [3](2004), J. Adell and P. Jodra [2](2008) and S. Koumandos [6](2013) for their surveys of other previous results on the Ramanujan sequence.

## 2. The results

### 2.1. More monotonicity properties

In this paper we refine the property (1.4) as follows.

**THEOREM 2.1.** *There is a probability distribution function  $G$  on  $[0, 1]$  such that*

$$(n+1)(\theta_n - 1/3) = \frac{4}{135} + \frac{37}{270} \int_0^1 x^n dG(x), \quad n \geq 0. \quad (2.1)$$

*As a consequence, the sequence*

$$\left\{ (n+1)(\theta_n - 1/3) \right\}_{n \geq 0}$$

*is completely monotone.*

We also show in Section 4 that Theorem 3.1 below yields easily (1.4) and the validity of (1.6) written in the following equivalent form

$$\frac{4}{135} - n \cdot \left( \theta_n - \frac{1}{3} \right) = \frac{4}{135} \int_0^1 t^n dD(t), \quad n = 0, 1, 2, \dots, \quad (2.2)$$

where  $D$  is a continuous probability distribution function on  $[0, 1]$ .

### 2.2. The Ramanujan sequence and univalent functions

Let  $\Lambda$  denote the slit domain  $\mathbb{C} \setminus [1, \infty)$  and  $\text{Hol}(\Lambda)$  the set of analytic functions in  $\Lambda$ . We write  $f \in \text{Hol}_1(\Lambda)$  if  $f \in \text{Hol}(\Lambda)$  satisfies  $f(0) = f'(0) - 1 = 0$ . The following definition has been introduced in [10, Def.1.3, 1.4, pp. 290–291].

**DEFINITION 1.** A function  $f \in \text{Hol}_1(\Lambda)$  is called universally starlike if it maps every circular domain  $\Omega \subset \Lambda$  with  $0 \in \Omega$  conformally onto a domain starlike with respect to the origin. It is called universally convex if it maps every circular domain  $\Omega \subset \Lambda$  conformally onto a convex domain.

Note that in this definition circular domains are meant to be open disks or open half-planes in  $\mathbb{C}$ . It is an immediate consequence of the definition that

$$f \text{ universally convex} \Rightarrow f \text{ universally starlike} . \quad (2.3)$$

In [10, p.294] the following conjecture has been proposed.

CONJECTURE A (S. Ruscheweyh, L. Salinas, and T. Sugawa, 2009 [10]). *The function*

$$\sigma(z) := \frac{1}{\theta_1 - 1/3} \cdot \sum_{n=1}^{\infty} (\theta_n - 1/3) \cdot z^n \quad (2.4)$$

*is universally convex.*

In a recent paper [4] the present authors established the following general result.

THEOREM A. *For  $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \text{Hol}(\Lambda)$  let  $f_{\alpha}(z) := \sum_{n=1}^{\infty} n^{-\alpha} a_n z^n$ . Then we have:*

1. *If  $f$  is universally convex then the functions  $f_{\alpha}$ ,  $\alpha \geq 1$ , are also universally convex.*
2. *If  $f$  is universally starlike then the functions  $f_{\alpha}$ ,  $\alpha \geq 0$ , are also universally starlike.*

We shall prove

THEOREM 2.2. *The functions*

$$\sigma_{\alpha}(z) := \frac{1}{\theta_1 - 1/3} \sum_{n=1}^{\infty} (\theta_n - 1/3) \frac{z^n}{n^{\alpha}} ,$$

*are universally starlike for every  $\alpha \geq 1$ .*

In view of Theorem A and (2.3) it is clear that Theorem 2.2 represents a necessary condition for Conjecture A to be valid, and therefore is a first step towards the still open decision concerning Conjecture A.

### 3. Watson's approach

In this section we follow Watson's reasonings from [12]. On the positive half-line there exist two functions  $u$  and  $U$  satisfying the following relations

$$u(x) \cdot e^{1-u(x)} = e^{-x} , \quad U(x) \cdot e^{1-U(x)} = e^{-x} , \quad 0 \leq u(x) \leq 1 \leq U(x) , \quad x \geq 0 . \quad (3.1)$$

The function  $U$  is strictly increasing on  $[0, +\infty)$  with  $U(0) = 1$  and  $U(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$ , whereas  $u$  is strictly decreasing on  $[0, +\infty)$  with  $u(0) = 1$  and  $\lim_{x \rightarrow +\infty} u(x) = 0$ . Furthermore,  $u$  and  $U$  satisfy the differential equations (see [12, p.295])

$$U'(x) = \frac{U(x)}{U(x) - 1} , \quad u'(x) = -\frac{u(x)}{1 - u(x)} , \quad x > 0 , \quad (3.2)$$

which imply that

$$\begin{aligned} U''(x) &= -\frac{U(x)}{(U(x)-1)^3}, & u''(x) &= \frac{u(x)}{(1-u(x))^3}, \\ U'''(x) &= \frac{U(x)(2U(x)+1)}{(U(x)-1)^5}, & u'''(x) &= -\frac{u(x)(2u(x)+1)}{(1-u(x))^5}. \end{aligned} \quad (3.3)$$

If we put

$$w(x) := u(\log x), \quad W(x) := U(\log x), \quad x \geq 1, \quad (3.4)$$

then

$$\frac{e^{w(x)}}{e \cdot w(x)} = x, \quad \frac{e^{W(x)}}{e \cdot W(x)} = x, \quad 0 \leq w(x) \leq 1 \leq W(x), \quad x \geq 1, \quad (3.5)$$

and therefore

$$w\left(\frac{e^x}{e \cdot x}\right) = x, \quad x \in [0, 1], \quad W\left(\frac{e^x}{e \cdot x}\right) = x, \quad x \geq 1. \quad (3.6)$$

Since, on the positive semiaxis, the function  $x/(e^x - 1)$  decreases from 1 to zero while  $x/(1 - e^{-x})$  increases from 1 to  $+\infty$  the equations (3.6) can be written as

$$w\left(\frac{\exp\left(\frac{x}{e^x - 1}\right)}{e \cdot \frac{x}{e^x - 1}}\right) = \frac{x}{e^x - 1}, \quad W\left(\frac{\exp\left(\frac{x}{1 - e^{-x}}\right)}{e \cdot \frac{x}{1 - e^{-x}}}\right) = \frac{x}{1 - e^{-x}}, \quad x \geq 0. \quad (3.7)$$

Elementary calculations show that the even function

$$\rho(x) := \frac{\exp\left(\frac{x}{e^x - 1}\right)}{e \cdot \frac{x}{e^x - 1}} = \frac{\exp\left(\frac{x}{1 - e^{-x}}\right)}{e \cdot \frac{x}{1 - e^{-x}}}, \quad x \in \mathbb{R}, \quad (3.8)$$

satisfies  $\rho(0) = 1$  and

$$\frac{\rho'(x)}{\rho(x)} = \frac{e^x(e^x - 1 - x)(e^{-x} - 1 + x)}{x(e^x - 1)^2} > 0, \quad x > 0. \quad (3.9)$$

Therefore (3.8) and (3.4) imply

$$U(\log \rho(x)) = \frac{x}{1 - e^{-x}} =: H(x), \quad u(\log \rho(x)) = \frac{x}{e^x - 1} =: h(x), \quad x \in \mathbb{R}. \quad (3.10)$$

and by virtue of (3.3) we obtain for arbitrary  $x > 0$  the representations

$$U'(\log \rho(x)) + u'(\log \rho(x)) = \frac{H(x)}{H(x) - 1} + \frac{h(x)}{h(x) - 1}, \quad (3.11)$$

$$U''(\log \rho(x)) + u''(\log \rho(x)) = \frac{H(x)}{(1 - H(x))^3} + \frac{h(x)}{(1 - h(x))^3} \quad (3.12)$$

$$U'''(\log \rho(x)) + u'''(\log \rho(x)) = H(x) \frac{1 + 2H(x)}{(H(x) - 1)^5} + h(x) \frac{1 + 2h(x)}{(h(x) - 1)^5}, \quad (3.13)$$

The following key result concerning these quantities will be established in Section 6.

THEOREM 3.1. For  $x > 0$  we have

$$U'(x) + u'(x) > -(U''(x) + u''(x)) > U'''(x) + u'''(x) > 0. \quad (3.14)$$

The last inequality in (3.14) is known (see Koumandos [6, Lemma 2, p.452]). However, in order to make the present paper more self-contained, a new proof, based on the new algorithm disclosed in 6.1, is given in Subsection 6.2.

#### 4. Proof of Theorem 2.1

G.N.Watson [12, p.297] obtained

$$\theta_n - \frac{1}{3} = \frac{1}{2} \int_0^\infty e^{-nx} (-U''(x) - u''(x)) dx, \quad n \geq 0, \quad (4.1)$$

and [12, p.300]

$$U''(0) + u''(0) = -\frac{8}{135}. \quad (4.2)$$

Integration by parts applied to (4.1) using (4.2) gives the basic relations used for this proof:

$$n \left( \theta_n - \frac{1}{3} \right) = \frac{4}{135} - \frac{1}{2} \int_0^\infty e^{-nx} (U'''(x) + u'''(x)) dx, \quad n \geq 0, \quad (4.3)$$

$$\frac{\theta_n - \frac{1}{3}}{n} = \frac{1}{2} \int_0^\infty e^{-nx} \left( \frac{4}{3} - U'(x) - u'(x) \right) dx, \quad n \geq 1, \quad (4.4)$$

$$(n+1) \left( \theta_n - \frac{1}{3} \right) - \frac{4}{135} = \frac{1}{2} \int_0^\infty e^{-nx} \Delta(x) dx, \quad n \geq 0, \quad (4.5)$$

where

$$\Delta(x) := -U''(x) - u''(x) - U'''(x) - u'''(x), \quad x \geq 0. \quad (4.6)$$

We mention in passing that (4.4) leads immediately to the representation (1.4) for the Ramanujan sequence given by J. Adell and P. Jodra [2, (5), p.3]. It follows from (3.8), (3.9) and Theorem 3.1 that

$$\lim_{x \rightarrow +\infty} (u'(x) + U'(x)) = 1, \quad u'(0) + U'(0) = 4/3, \quad u'(x) + U'(x) > 0, \quad x > 0. \quad (4.7)$$

Theorem 3.1 (see also Watson [12, p.298] and Alzer [3, p.641]) implies  $U''(x) + u''(x) < 0$ ,  $x > 0$ , so that the function

$$G_0(x) := 3 \left[ \frac{4}{3} - U'(x) - u'(x) \right], \quad x \geq 0, \quad (4.8)$$

increases from 0 to 1 on the positive half-line and in view of (4.1),

$$\theta_n - \frac{1}{3} = \frac{1}{6} \int_0^\infty e^{-nx} dG_0(x) = \frac{1}{6} \int_0^1 t^n d \left[ 1 - G_0 \left( \log \frac{1}{t} \right) \right], \quad n \geq 0.$$

Since  $\theta_n > 0$ ,  $n \geq 0$ , and

$$\sum_{m=0}^n C_n^m (-1)^m \theta_{k+m} = \sum_{m=0}^n C_n^m (-1)^m \left( \theta_{k+m} - \frac{1}{3} \right) = \frac{1}{6} \int_0^\infty e^{-kx} (1 - e^{-x})^n dG_0(x) > 0,$$

for all  $k \geq 0$  and  $n \geq 1$ , we obtain the complete monotony of  $\{\theta_n\}_{n \geq 0}$  and the validity of (1.4) for  $F(x) := 1 - G_0(\log 1/x)$ ,  $0 < x \leq 1$ ,  $F(0) := 0$ .

Furthermore, by writing (4.3) in the form

$$1 - \frac{135n}{4} \left( \theta_n - \frac{1}{3} \right) = \frac{135}{8} \int_0^\infty e^{-nx} (U'''(x) + u'''(x)) dx = \int_0^\infty e^{-nx} dG_1(x), \quad n \geq 0,$$

where

$$G_1(x) = 1 + \frac{135}{8} (U''(x) + u''(x)), \quad x \geq 0,$$

we obtain the validity of (2.2) for  $D(x) := 1 - G_1(\log(1/x))$  because  $U''' + u'''$  is non-negative for all  $x \geq 0$  by Theorem 3.1 (see also Koumandos [6, p.452]).

Using (4.5) and again Theorem 3.1 to see that  $\Delta(x)$  is non-negative we conclude that

$$(n+1) \left( \theta_n - \frac{1}{3} \right) - \frac{4}{135} = \frac{37}{270} \int_{[0,1]} x^n dG(x), \quad n \geq 0,$$

where

$$\frac{37}{45} G(e^{-x}) := 3 [U'(x) + u'(x) + U''(x) + u''(x) - 1], \quad 0 \leq x < +\infty,$$

$G(0) := G(0+0) = 0$ ,  $G(1-0) = G(1) = 1$  and  $(37/45)e^{-x}G'(e^{-x}) = 3\Delta(x) > 0$  for all  $x > 0$ . Therefore  $G$  is a probability distribution function on  $[0, 1]$  and the sequence in question turns out to be completely monotone. The proof of Theorem 2.1 is now complete.

## 5. Proof of Theorem 2.2

Note that (4.1) and (4.4) imply the following integral representations for the functions dealt with in (2.4) and Theorem 2.2 (for  $\alpha = 1$ ),

$$(\theta_1 - 1/3) \sigma(z) = \frac{1}{2} \int_1^\infty \frac{z}{t-z} \frac{-U''(\log t) - u''(\log t)}{t} dt, \quad z \in \Lambda, \quad (5.1)$$

$$(\theta_1 - 1/3) \sigma_1(z) = \frac{1}{2} \int_1^\infty \frac{z}{t-z} \frac{(4/3) - U'(\log t) - u'(\log t)}{t} dt, \quad z \in \Lambda, \quad (5.2)$$

and recall, using Theorem A, that we need to prove Theorem 2.2 only for the case  $\alpha = 1$ . The necessary and sufficient condition for  $\sigma_1$  to have that property is given in the following result.

**THEOREM B** (Corollary 1.1 [10]). *Let  $f \in \text{Hol}_1(\Lambda)$ . Then  $f$  is universally starlike if and only if there exists a probability measure  $\mu$  on  $[0, 1]$  such that*

$$\frac{f(z)}{z} = \exp \int_{[0,1]} \log \frac{1}{1-tz} d\mu(t), \quad z \in \Lambda. \quad (5.3)$$

### 5.1. Auxiliary results

The following lemma is from [10, Theorem 1.10, p.294].

**LEMMA A.** *Let  $\varphi, \psi : (0, 1) \rightarrow [0, +\infty)$  be two integrable functions on  $[0, 1]$  satisfying*

$$\int_0^1 \varphi(t) dt = \int_0^1 \psi(t) dt > 0, \quad \left| \begin{array}{cc} \varphi(x_2) & \psi(x_2) \\ \varphi(x_1) & \psi(x_1) \end{array} \right| \geq 0, \quad 0 < x_1 \leq x_2 < 1. \quad (5.4)$$

Then there exists a probability measure  $\mu$  on  $[0, 1]$  such that

$$\int_{[0,1]} \frac{d\mu(t)}{1-tz} = \int_0^1 \frac{\varphi(t)}{1-tz} dt / \int_0^1 \frac{\psi(t)}{1-tz} dt, \quad z \in \Lambda. \quad (5.5)$$

Lemma A allows us to prove the next statement.

LEMMA 5.1. *Let  $g : (0, +\infty) \rightarrow (0, +\infty)$  be twice continuously differentiable on  $(0, \infty)$  and assume*

$$\begin{aligned} (a) \quad & \lim_{x \downarrow 0} g(x) = 0, \quad (b) \quad \int_0^\infty e^{-x} g(x) dx = 1, \\ (c) \quad & g'(x) \geq 0, \quad x > 0, \quad (d) \quad g'(x)^2 - g''(x)g(x) \geq 0, \quad x > 0. \end{aligned} \quad (5.6)$$

Then there exists a probability measure  $\mu$  on  $[0, 1]$  such that

$$\int_1^\infty \frac{g(\log t)/t}{t-z} dt = \exp \int_{[0,1]} \log \frac{1}{1-tz} d\mu(t), \quad z \in \Lambda. \quad (5.7)$$

*Proof.* Denote  $v(x) := g(\log x)/x$ ,  $x > 1$ , and for  $z \in \Lambda$  let

$$f(z) := z \int_1^\infty \frac{g(\log t)/t}{t-z} dt = z \int_1^\infty \frac{v(t)}{t-z} dt = \int_1^\infty \left[ -1 + \frac{t}{t-z} \right] v(t) dt.$$

The properties (5.6)(a),(b) mean that  $v \in L_1([1, +\infty))$  and  $\lim_{x \rightarrow 1+0} v(x) = 0$ , which implies that for arbitrary  $z \in \Lambda$  we have

$$f'(z) = \int_1^\infty \frac{tv(t)}{(t-z)^2} dt = - \int_1^\infty tv(t) d \frac{1}{t-z} = \frac{v(1)}{1-z} + \int_1^\infty \frac{(tv(t))'}{t-z} dt = \int_1^\infty \frac{(tv(t))'}{t-z} dt,$$

and

$$\frac{zf'(z)}{f(z)} = \frac{\int_1^\infty \frac{(tv(t))'}{t-z} dt}{\int_1^\infty \frac{v(t)}{t-z} dt} = \frac{\int_0^1 \frac{[(1/t)v'(1/t) + v(1/t)]/t}{1-tz} dt}{\int_0^1 \frac{v(1/t)/t}{1-tz} dt} = \frac{\int_0^1 \frac{\varphi(t)}{1-tz} dt}{\int_0^1 \frac{\psi(t)}{1-tz} dt},$$

where

$$\varphi(x) := ((1/x)v'(1/x) + v(1/x))/x, \quad \psi(x) := v(1/x)/x, \quad 0 < x < 1,$$

and

$$\int_0^1 \varphi(t) dt = \int_0^1 \psi(t) dt = \int_1^\infty \frac{v(x)}{x} dx = \int_1^\infty \frac{g(\log x)}{x^2} dx = \int_0^\infty \frac{g(x)}{e^x} dx = 1.$$

Moreover, it follows from (5.6)(d) that the function  $g'(\log x)/g(\log x)$  is non-increasing on  $(1, +\infty)$  and since

$$1 + \frac{xv'(x)}{v(x)} = \frac{\frac{d}{dx}[xv(x)]}{v(x)} = \frac{\frac{d}{dx} \left[ x \cdot \frac{g(\log x)}{x} \right]}{\frac{g(\log x)}{x}} = \frac{\frac{g'(\log x)}{x}}{\frac{g(\log x)}{x}} = \frac{g'(\log x)}{g(\log x)}$$

the function  $xv'(x)/v(x)$  also does not increase on  $(1, +\infty)$ . This means that for arbitrary  $0 < x_1 \leq x_2 < 1$  we have

$$0 \leq \frac{1}{x_1 x_2} \begin{vmatrix} v(1/x_1) & (1/x_1)v'(1/x_1) \\ v(1/x_2) & (1/x_2)v'(1/x_2) \end{vmatrix} = \begin{vmatrix} v(1/x_1)/x_1 & (1/x_1^2)v'(1/x_1) \\ v(1/x_2)/x_2 & (1/x_2^2)v'(1/x_2) \end{vmatrix} = \begin{vmatrix} \psi(x_1) & \varphi(x_1) \\ \psi(x_2) & \varphi(x_2) \end{vmatrix}.$$

Lemma A guarantees the existence of a probability measure  $\mu$  on  $[0, 1]$  such that

$$\frac{zf'(z)}{f(z)} = \int_{[0,1]} \frac{d\mu(t)}{1-tz}, \quad z \in \Lambda.$$

Since

$$\frac{\frac{d}{dz} \frac{f(z)}{z}}{\frac{f(z)}{z}} = \frac{\frac{zf'(z) - f(z)}{z^2}}{\frac{f(z)}{z}} = \frac{f'(z)}{f(z)} - \frac{1}{z} = \int_{[0,1]} \frac{d}{dz} \log \frac{1}{1-tz} d\mu(t)$$

we can integrate this equality from 0 to  $z \in \Lambda$  and obtain

$$\log \frac{f(z)}{z} - \log f'(0) = \int_{[0,1]} \log \frac{1}{1-tz} d\mu(t),$$

where  $f'(0) = 1$  by virtue of (5.6)(b). Lemma 5.1 is proved.  $\square$

## 5.2. The proof

By Theorem B the statement of Theorem 2.2 for  $\alpha = 1$  means that for  $\sigma_1$  there exists a probability measure  $\mu$  on  $[0, 1]$  such that

$$\frac{\sigma_1(z)}{z} = \exp \int_{[0,1]} \log \frac{1}{1-tz} d\mu(t), \quad z \in \Lambda, \quad (5.8)$$

where in accordance with (5.2),

$$2(\theta_1 - 1/3) \frac{\sigma_1(z)}{z} = \int_1^\infty \frac{1}{t-z} \frac{g(\log t)}{t} dt, \quad z \in \Lambda, \\ g(x) := \frac{4}{3} - U'(x) - u'(x), \quad x > 0,$$

and in view of (4.4)

$$\int_0^\infty e^{-x} g(x) dx = \int_0^\infty e^{-x} [4/3 - U'(x) - u'(x)] dx = 2(\theta_1 - 1/3).$$

Moreover, Theorem 3.1 and (4.7) imply that for arbitrary  $x > 0$

$$g(x) = \frac{4}{3} - U'(x) - u'(x) \in \left[0, \frac{1}{3}\right], \quad g'(x) = -U''(x) - u''(x) > 0, \\ g''(x) = -U'''(x) - u'''(x) < 0, \quad g(0) = \frac{4}{3} - U'(0) - u'(0) = 0.$$

Thus,

$$g'(x)^2 - g''(x)g(x) > 0, \quad x > 0,$$

and Lemma 5.1 yields the validity of (5.8).



## 6. Proof of Theorem 3.1

### 6.1. An algorithm

In this section we present a general algorithm which gives sufficient conditions for inequalities of the type described in Theorem 3.1. It is dealing with exponential polynomials on  $\mathbb{R}^+$ .

DEFINITION 2. Let

$$f(x) := \sum_{k=0}^m P_k(x)e^{kx},$$

where the  $P_k$  are real polynomials of exact degree  $n_k$ . Then we call  $f$  an *exponential polynomial* of order  $m$  and (multi-)degree  $\{n_0, \dots, n_m\}$ .

REMARK 1. A polynomial  $P(x) = \sum_{j=0}^n a_j x^j$  is said to be of exact degree  $n$  if  $a_n \neq 0$ . If  $P \equiv 0$  then we say it is of (exact) degree  $-1$ .

THEOREM 6.1. Let

$$f(x) := f_0(x) = \sum_{k=0}^m P_k(x)e^{kx} = \sum_{k=0}^{\infty} a_k x^k$$

be an exponential polynomial of order  $m$  and degree  $\{n_0, \dots, n_m\}$ . Let

$$f_{k+1}(x) := f_k^{(n_k+1)}(x)e^{-x}, \quad k = 0, \dots, m-1, \quad (6.1)$$

and assume that

$$f_k^{(s)}(0) \geq 0, \quad s = 0, \dots, n_k, \quad k = 0, \dots, m. \quad (6.2)$$

Then all Taylor coefficients  $a_k$ ,  $k \geq 0$ , of  $f(x)$  are non-negative. In particular,  $f(x) \geq 0$ ,  $x \geq 0$ .

REMARK 2. Note that there are only *finitely* many, namely

$$\mu(f) := \sum_{k=0}^m (n_k + 1),$$

conditions to be tested (which involve only the first  $\mu(f)$  coefficients  $a_k$  of  $f$ ) to draw the conclusion for *all* coefficients of  $f$ .

*Proof.* The proof runs by mathematical induction. First note that for any exponential polynomial

$$h(x) = \sum_{k=0}^m P_k(x)e^{kx}, \quad \text{degree}(h) = \{n_0, \dots, n_m\},$$

we have

$$h'(x) = P'_0(x) + \sum_{k=1}^m (P'_k(x) + kP_k(x))e^{kx},$$

which is an exponential polynomial of order  $m$  and degree  $\{n_0 - 1, n_1, \dots, n_m\}$ . Further, if  $n_0 = 0$ , the function  $h'(x)e^{-x}$  is an exponential polynomial of order  $m - 1$  and degree  $\{n_1, \dots, n_m\}$ .

We begin with the case  $m = 0$ . Then we have  $f = P_0$  with degree  $\{n_0\}$ . In this case the conditions (6.2) just say

$$P_0^{(s)}(0) \geq 0, \quad s = 0, \dots, n_0,$$

which means that all coefficients of  $P_0$  (and therefore of  $f$ ) are non-negative. This settles the case  $m = 0$ .

Now assume that the theorem is valid for some  $m - 1 \geq 0$ , and let  $f = f_0$  be as in the statement of the theorem. The way the function  $f_1$  is defined it is clear that it is an exponential polynomial of order  $m - 1$ , and degree  $\{n_1, \dots, n_m\}$  and the conditions (6.2), applied to  $f_1$  instead of  $f_0$ , show, by our assumption that the theorem is correct for functions of degree  $m - 1$ , that  $f_1$  has all of its Taylor coefficients non-negative, which implies that the Taylor coefficients of

$$f_0^{(n_0+1)}(x) = e^x f_1(x) \tag{6.3}$$

are also all non-negative. The conditions (6.2) concerning  $f_0$  now say that the remaining first coefficients of  $f$ , namely  $a_s = f_0^{(s)}(0)/s!$ ,  $s = 0, \dots, n_0$ , are non-negative as well. This completes the proof.  $\square$

When it comes to the application of this theorem we have to keep the following facts in mind:

- (1) This algorithm is particularly suited for cases when the polynomials  $P_k$  have rational coefficients only since then all the numbers  $f_k^{(s)}(0)$  are rational and therefore their calculation via a computer algebra program is exact and no numerical problem, f.i. with cancelation, occurs.
- (2) Given an exponential polynomial it is not really necessary to know its order or multi-degree to begin with: the algorithm can decide, when properly implemented, by itself what to do next (differentiate once more or go to the next  $f_k$ ). In particular it can stop as soon as one of the numbers  $f_k^{(s)}(0)$  turns out to be negative, which can save machine time.

## 6.2. The proof

The proof of Theorem 3.1 will be completely computer based, using the algorithm just described. All coefficients in these cases are rational, actually integers, so there is no numerical problem. The algorithm has been programmed using Mathematica version 9.0 and run on a laptop computer. Computation time was a few seconds for each of the three cases to be verified for Theorem 3.1.

The resulting numbers  $f_k^{(s)}(0)$  are collected in one single vector  $\lambda(f)$  with  $\mu(f)$  entries, listed in their natural order as they are being calculated by the algorithm. If  $\lambda(f)$  turns out to be non-negative then the case under consideration has been settled.

### 6.2.1. Case 1: $(\mathbf{U}' + \mathbf{u}') + (\mathbf{U}'' + \mathbf{u}'') > \mathbf{0}$ .

Using (3.11), (3.12) we find for

$$\begin{aligned} R_1(x) &:= \frac{H(x)}{H(x) - 1} + \frac{h(x)}{h(x) - 1} - \frac{H(x)}{(H(x) - 1)^3} - \frac{h(x)}{(h(x) - 1)^3} \\ &= \frac{x^2}{((e^x - 1 - x)^3(1 - e^x(1 - x)))^3} S_1(x), \end{aligned}$$

where

$$S_1(x) = (-2 - x) + (8 - 3x - 3x^2)e^x + (-14 + 9x - 6x^2 - 5x^3)e^{2x}$$

$$\begin{aligned}
& + (16 + 18x^2 - 2x^4)e^{3x} + (-14 - 9x - 6x^2 + 5x^3)e^{4x} \\
& + (8 + 3x - 3x^2)e^{5x} + (-2 + x)e^{6x}.
\end{aligned}$$

So  $S_1$  is an exponential polynomial of order 6 and degree  $\{1, 2, 3, 4, 3, 2, 1\}$ . Application of the algorithm produces the vector

$$\begin{aligned}
\lambda(S_1) = & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 72240, 1155840, 9557760, 56267040, \\
& 271084224, 880843680, 2475629568, 6343909632, \\
& 1533939393792, 20392197120, 25057382400, 29561241600, \\
& 4478976000),
\end{aligned}$$

which proves that the desired inequality is valid.

### 6.2.2. Case 2: $-(\mathbf{U}'' + \mathbf{u}'') - (\mathbf{U}''' + \mathbf{u}''') > \mathbf{0}$ .

Here we have to show that

$$\begin{aligned}
R_2(x) &:= \frac{H(x)}{(H(x) - 1)^3} + \frac{h(x)}{(h(x) - 1)^3} - H(x) \frac{1 + 2H(x)}{(H(x) - 1)^5} - h(x) \frac{1 + 2h(x)}{(h(x) - 1)^5} \\
&= \frac{x^2(e^x - 1)^2}{((e^x - 1 - x)^5(1 - e^x(1 - x))^5)} S_2(x),
\end{aligned}$$

where

$$\begin{aligned}
S_2(x) = & (-4 - x) + (24 - 15x - 5x^2)e^x + (-64 + 70x - 60x^2 - 50x^3 - 20x^4 - 4x^5)e^{2x} \\
& + (104 - 91x + 285x^2 + 100x^3 + 20x^4 - x^5 - x^6)e^{3x} + (-120 - 440x^2)e^{4x} \\
& + (104 + 91x + 285x^2 - 100x^3 + 20x^4 + x^5 - x^6)e^{5x} \\
& + (-64 - 70x - 60x^2 + 50x^3 - 20x^4 + 4x^5)e^{6x} + (24 + 15x - 5x^2)e^{7x} + (-4 + x)e^{8x}.
\end{aligned}$$

So  $S_2$  is an exponential polynomial of order 8 and degree  $\{1, 2, 5, 6, 2, 6, 5, 2, 1\}$ . Application of the algorithm produces the vector

$$\begin{aligned}
\lambda(S_2) = & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1095494400, 38342304000, \\
& 718413696000, 8922167654400, 85789518796800, 686634000998400, \\
& 4108040955648000, 21277519458048000, 98491821821245440, \\
& 417993857883463680, 1659729058910208000, 6264125727645450240, \\
& 22744955668622376960, 57435249160046592000, 138673044884876820480, \\
& 324272107555238707200, 741041088684097536000, 1665009811944898560000, \\
& 3693054970331136000000, 4415481367363584000000, 5133351192625152000000, \\
& 5850215720681472000000, 716770887598080000000)
\end{aligned}$$

which proves the claim as all entries are non-negative.

### 6.2.3. Case 3: $(\mathbf{U}''' + \mathbf{u}''') > \mathbf{0}$ .

Here we have to show that

$$R_3(x) := H(x) \frac{1 + 2H(x)}{(H(x) - 1)^5} + h(x) \frac{1 + 2h(x)}{(h(x) - 1)^5} = \frac{x(e^x - 1)^3}{((e^x - 1 - x)^5(1 - e^x(1 - x))^5)} S_3(x) > 0,$$

where

$$\begin{aligned} S_3(x) := & 1 - 2x + (-5 + 20x + 10x^3 + 5x^4 + x^5)e^x \\ & + (9 - 72x - 70x^3 - 30x^4 - 11x^5 - 2x^6)e^{2x} + (-5 + 130x + 160x^3 + 25x^4 - 10x^5)e^{3x} \\ & + (-5 - 130x - 160x^3 + 25x^4 - 10x^5)e^{4x} + (9 + 72x + 70x^3 - 30x^4 + 11x^5 - 2x^6)e^{5x} \\ & + (-5 - 20x - 10x^3 + 5x^4 - x^5)e^{6x} + (1 + 2x)e^{7x}. \end{aligned}$$

So  $S_3$  is an exponential polynomial of order 7 and degree  $\{1, 5, 6, 5, 5, 6, 5, 1\}$ . Application of the algorithm produces the vector

$$\begin{aligned} \lambda(S_3) = & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 115315200, 3863059200, 70457587200, \\ & 927826099200, 9830767564800, 8631514316800, 615374090956800, \\ & 83729093713049600, 20168695176376320, 99183876729477120, 450524284521338880, \\ & 1915432618475059200, 5792081300977213440, 16127157987099279360, \\ & 41953781738132766720, 103330763975294484480, 243753521061983846400, \\ & 556095351762151833600, 1236678576792676761600, 1461058224846520320000, \\ & 1642128742165708800000, 1795208480980992000000, 1936007205617664000000, \\ & 2073220384555008000000, 2209799770472448000000, 1365277881139200000000) \end{aligned}$$

which is also non-negative. The proof is complete.

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